FROM REAL AND IMAGINARY TO LINEAR AND ORTHOGONAL: A PEDAGOGICAL APPROACH TO COMPLEX NUMBERS

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ABSTRACT

The term imaginary number is historically established but pedagogically problematic. In ordinary language, imaginary denotes something fictitious or nonexistent, whereas imaginary and complex numbers are mathematically rigorous and indispensable in engineering, physics, signal processing and other scientific disciplines. This semantic conflict may contribute to students' perception of complex numbers as artificial, mysterious or less legitimate than real numbers. This article examines the historical development of the terminology, analyses its possible cognitive consequences and evaluates proposed alternatives such as vertical, orthogonal, rotational, phase and normal numbers. It argues that formally replacing the standard terminology would create new mathematical ambiguities and disrupt continuity across the literature. Instead, a dual-language pedagogical framework is proposed: teachers should retain the conventional terms real and imaginary while initially describing their geometric and operational meanings through expressions such as reference-axis component, orthogonal component and quarter-turn operator. In this framework, a complex number **z=a+bi** is introduced as a unified two-component mathematical object, while multiplication by **i** is interpreted as a **90^0** rotation in the complex plane. This approach respects mathematical history and formal precision while reducing the misconceptions generated by inherited terminology.

Keywords: complex numbers; imaginary numbers; mathematical terminology; mathematics education; conceptual understanding; visualisation; complex plane

1. INTRODUCTION

Mathematical terminology does more than label concepts. It shapes the preliminary mental images through which learners interpret them. When the everyday meaning of a mathematical term conflicts with its technical meaning, students may develop intuitions that obstruct rather than support formal understanding.

The expression imaginary number provides a particularly important example. In ordinary English, imaginary generally means fictitious, invented or existing only in the mind. A learner encountering the term for the first time may therefore conclude that imaginary numbers are somehow unreal, invalid or merely symbolic devices. Mathematically, however, imaginary numbers are no less legitimate than negative, irrational or real numbers. They form part of the complex number system and are essential in electrical engineering, wave analysis, control systems, quantum mechanics, signal processing and many other fields.

Research in mathematics education indicates that learners often regard complex numbers as artificial constructions, disconnected combinations of real and imaginary pieces, symbolic tricks involving $\mathbf{i=\sqrt{-1}}$, or concepts that cannot be visualised meaningfully. Some students also fail to recognise that every real number is itself a complex number with zero imaginary component (Cortas Nordlander and Nordlander, 2012). Studies involving university students and prospective physics teachers similarly report conceptual obstacles concerning the nature, representation and operations of complex numbers (Mutambara and Tsakeni, 2022).

The central question is therefore not whether the established mathematical terminology should simply be discarded. Such a proposal would be neither realistic nor necessarily desirable. The more productive question is:

> Can complex numbers be introduced through more descriptive language without sacrificing established terminology or mathematical accuracy?

This article argues that they can. It proposes a pedagogical distinction between formal terminology, which should be retained, and descriptive bridge terminology, which can help learners construct an accurate geometric and operational understanding.

2. Historical Background

The quantities now called imaginary numbers emerged from attempts to solve polynomial equations. During the sixteenth century, expressions involving square roots of negative quantities appeared in solutions of cubic equations. Although these expressions were initially treated with suspicion, mathematicians such as Rafael Bombelli demonstrated that they could be manipulated according to coherent algebraic rules.

René Descartes used the term imaginary in La Géométrie, published in 1637, while discussing roots that did not correspond to what he considered admissible geometric solutions. The terminology reflected the uncertain mathematical status of such roots at that historical stage rather than the modern understanding of complex numbers as well-defined mathematical objects (Descartes, 1637/1954; Nahin, 1998).

The term remained, even as the concept evolved. By the nineteenth century, geometric interpretations associated complex numbers with points or directed segments in a plane. Gauss argued that much of the obscurity surrounding the subject resulted from unsuitable terminology. He suggested that **+1, -1** and **√-1** might have been understood more readily had they been described as direct, inverse and lateral units rather than positive, negative and imaginary units (Gauss, 1831/1832). Gauss also emphasised that real numbers should not be viewed as opposites of complex numbers, since the real numbers are contained within the complex number system.

In 1936, Arnold Dresden proposed the expression normal numbers, using normal in its geometric sense of perpendicular. The proposed terminology was intended to connect imaginary quantities with the axis perpendicular to the real axis (Dresden, 1936; Kramer, 1981). The suggestion did not gain widespread acceptance.

These historical proposals reveal that dissatisfaction with the term imaginary is not a recent popular-science phenomenon. Mathematicians themselves have long recognised that the terminology can obscure the concept it is intended to represent.

3. What is mathematically misleading about "imaginary"?

An imaginary number is ordinarily understood as a number of the form **bi**, **b€ R**, where $i^2=-1$

A general complex number has the form **z=a+bi, a,b€ R**

The quantities **a** and **b** are called the real and imaginary parts of **z,** where **a=Re(z), b=Im(z).**

Strictly speaking, the imaginary part is the real coefficient **b**, not the entire term **bi**. This distinction is itself frequently blurred in informal teaching.

The word imaginary becomes misleading when it is interpreted ontologically—as though it distinguishes existing numbers from nonexistent ones.

Mathematics does not classify numbers according to physical existence.

Natural, negative, irrational, real and complex numbers are components of formal mathematical systems defined by their properties and relationships.

Furthermore,

$\mathbf{R} \subset \mathbf{C}$

Every real number **a** may be written as **a+0i**.

Thus real and complex are not mutually exclusive classifications. Similarly, zero lies on both the real and imaginary axes because**0=0+0i**.

Describing a complex number as the combination of a "real thing" and an "imaginary thing" may therefore encourage a false division within a single mathematical object. Research on students' concept images suggests that learners often treat **a+bi** as two loosely attached numbers rather than one unified complex number (Cortas Nordlander and Nordlander, 2012).

4. THE GEOMETRIC MEANING OF COMPLEX NUMBERS

The most effective antidote to the linguistic problem is not merely a new name but a richer representation.

A complex number

$z=a+bi$ can be represented by the ordered pair **(a,b)** in the complex plane. The horizontal coordinate records the real component, while the vertical coordinate records the coefficient of the imaginary component.

This representation allows complex addition to be interpreted geometrically:

$$(a+bi)+(c+di)=(a+c)+(b+d)i.$$

Equivalently,

$$(a,b)+(c,d)=(a+c,b+d).$$

More importantly, multiplication by **i** has a direct geometric interpretation:

$$i(a+bi)=ai+bi^2=-b+ai.$$

Therefore, under multiplication by i,

$(a,b)\longmapsto(-b,a).(a,b)$

$$\begin{pmatrix} a \\ b \end{pmatrix} \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

is a counter-clockwise rotation through 90∘. Consequently,

$1\longmapsto i,\ i\longmapsto -1,\ -1\longmapsto -i,\ -i\longmapsto 1.$

This rotational interpretation gives iii an intelligible operational meaning. It is not simply a mysterious symbol declared to satisfy $i^2=-1$; it is an algebraic operator representing a quarter-turn in the plane.

Dynamic and visual representations can improve students' ability to connect the algebraic, rectangular and polar forms of complex numbers. Research using GeoGebra, for example, indicates that carefully designed visual activities can strengthen both conceptual and procedural understanding (Seloane, Ramaila and Ndlovu, 2023).

5. Are "Linear" and "Vertical" Better Names?

The terms linear number and vertical number have intuitive appeal, particularly when used with a diagram. They immediately direct attention towards the two perpendicular axes of the complex plane. However, they should not be presented as formal replacements without qualification.

5.1 Linear Number

Calling real numbers linear numbers may communicate that they can be represented on a number line. Nevertheless, the expression is mathematically ambiguous. The word linear already has extensive meanings in linear algebra, linear transformations, linear equations and vector spaces. Moreover, the complex numbers themselves form a vector space: they are two-dimensional over **R** but one-dimensional over **C.**

It is therefore more accurate to describe the real axis as the horizontal reference axis or one-dimensional real-number line than to rename real numbers categorically as linear numbers.

5.2 Vertical Number

The term vertical number offers an immediate visual association with the conventional complex-plane diagram. However, verticality is a feature of the chosen coordinate representation, not an intrinsic algebraic property. A diagram may be rotated without changing the underlying mathematics. The imaginary axis is intrinsically perpendicular to the real axis, but it is vertical only by convention.

Accordingly, vertical component may be useful in an introductory diagram, but orthogonal component is mathematically more precise.

5.3 Rotational Number

The expression rotational number captures one of the most important meanings of **i.** Multiplication by **i** rotates every complex number through $\mathbf{90^0}$. Multiplication by a general unit complex number

$\mathbf{e^{i\theta}=Cos\theta=iSin\theta}$ rotates a point through the angle θ.

Nevertheless, the term is not a complete replacement for imaginary number. A pure imaginary number such as **3i** is a point on the imaginary axis, while rotation is associated specifically with multiplication by **i** or by a complex number of modulus one. Thus it is more accurate to call **i** a quarter-turn operator than to call every number **bi** a rotational number.

5.4 Phase number

Complex numbers are widely used to represent magnitude and phase in waves, alternating-current circuits, signal processing and quantum theory. In polar form,

$$\mathbf{z=re^{i\theta}}$$

the modulus **r** represents magnitude and the argument θ represents phase or direction.

However, phase is a property of a general non-zero complex number, not exclusively of its imaginary component. Calling imaginary numbers phase numbers would therefore be useful only within particular applied contexts.

5.5 Normal number

Dresden's proposed term normal number had a defensible geometric motivation because normal can mean perpendicular. It is no longer a viable replacement, however, because normal number now has a well-established and entirely different meaning in number theory, associated with the distribution of digits in numerical expansions following the work of Borel (1909). Reusing the term would replace one ambiguity with another.

6. A dual-language pedagogical framework

The preceding analysis suggests that no single alternative term is sufficiently precise, universal and free from established mathematical meanings. The solution should therefore not be formal renaming but pedagogical supplementation.

A dual-language framework can introduce the standard terminology and its descriptive interpretation simultaneously.

A teacher might begin as follows:

> A complex number has two real coordinates. The first is measured along the real or reference axis. The second is measured along an axis perpendicular to it and is written using the unit i. This perpendicular component is historically called the imaginary component, although it is not fictitious or mathematically unreal.

Within this framework, the following correspondences may be used:

real axis⟷ reference or horizontal axis,
imaginary axis⟷ orthogonal or conventionally vertical axis,

and **i**⟷ imaginary unit and quarter-turn operator..

This approach has several advantages.

First, it preserves the formal vocabulary that students will encounter in textbooks, examinations, university courses and scientific literature.
Second, it directly addresses the misconception that imaginary means nonexistent.
Third, it links symbolic procedures with geometric transformations.
Fourth, it avoids presenting the real and imaginary parts as unequal in legitimacy.
Finally, it prepares students for polar form, Euler's formula, phasors, waves and rotations.

Veith and Bitzenbauer (2021) caution that imprecise definitions and ways of speaking can communicate mathematically incorrect conceptions of both the complex unit and angles. Their argument supports a pedagogical approach in which accessible language is used, but is explicitly connected to formal definitions rather than substituted for them.

7. A PROPOSED INSTRUCTIONAL SEQUENCE

An effective introduction to complex numbers may proceed through five conceptual stages.

7.1 BEGIN WITH CLOSURE

Students should first see why an extension of the real numbers is useful. The equation

$$X^2 + 1 = 0$$

has no solution in **R**. The number system is therefore extended by introducing an element **i** satisfying $i^2=-1$.

This should be compared with earlier extensions from natural numbers to integers, integers to rational numbers and rational numbers to real numbers. The complex numbers are not an arbitrary exception; they are another systematic extension of a number system.

7.2 PRESENT a+bi AS ONE NUMBER

Learners should be discouraged from regarding **a+bi** as unrelated fragments. The ordered-pair representation

$$a+bi \leftrightarrow (a,b)$$

should be introduced early.

7.3 USE THE COMPLEX PLANE IMMEDIATELY

The geometric representation should not be postponed until after extensive symbolic manipulation. Students should locate real, imaginary and general complex numbers in the same plane and observe that the real axis is contained within the complex plane.

7.4 Demonstrate multiplication as transformation

Multiplication by **i** should be shown as

$$(a,b) \longmapsto (-b,a)$$

Students can then understand $\mathbf{i^2=-1}$ geometrically: two quarter-turns produce a half-turn.

7.5 Connect the representation to applications

Only after the mathematical structure is secure should applications such as alternating-current circuits, oscillations, wave phase, signal analysis and quantum amplitudes be introduced. Applications demonstrate usefulness, but they should reinforce rather than replace mathematical understanding.

8. Interpreting the accompanying figure

Figure 1 may be used as an introductory visual comparison between the real and imaginary axes.

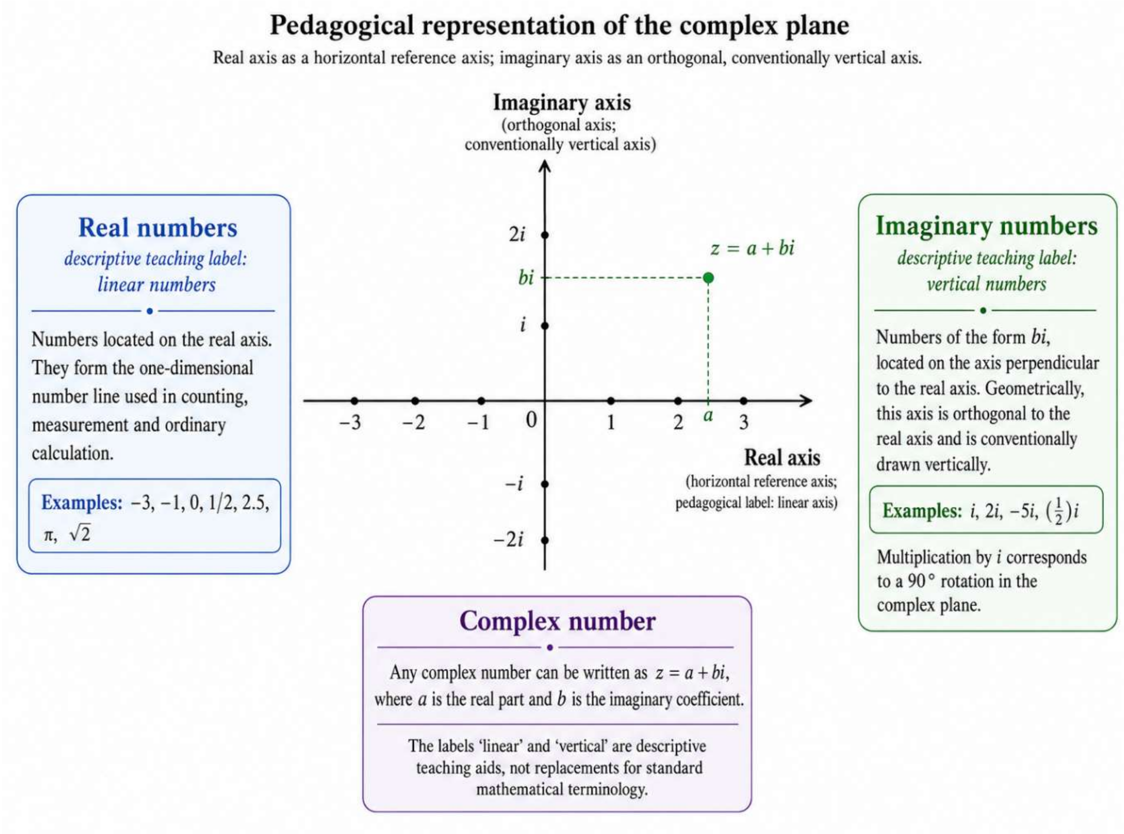

Figure 1. A pedagogical representation of the real axis as a horizontal reference axis and the imaginary axis as an orthogonal, conventionally vertical axis. The labels “linear” and “vertical” are descriptive teaching aids rather than replacements for standard mathematical terminology.

For journal publication, one sentence in the figure requires qualification. Rather than stating that “numbers have two dimensions”, the mathematically precise statement is:
The complex number system is two-dimensional when regarded as a vector space over the real numbers.
The figure should also make clear that **a** and **b** in **z=a+bi** are real numbers, while **bi** is the imaginary component. The label vertical number should be accompanied by a note explaining that verticality is diagrammatic, whereas perpendicularity is the underlying geometric relationship.

9. DISCUSSION

The persistence of traditional terminology is not merely evidence of mathematical conservatism. Standardised vocabulary allows results to be communicated across languages, institutions, disciplines and generations. Replacing real and imaginary throughout mathematical literature would impose substantial costs while potentially creating new ambiguities.
The case for reform is therefore strongest at the level of explanation rather than official nomenclature. Historical terms can remain while teachers expose their limitations. Comparable approaches are already common elsewhere in mathematics: an irrational number is explained as a number that is not expressible as a ratio, not as an unreasonable number; a negative number is not treated as a defective number; and a complex number need not be psychologically complicated.
The real educational objective is to ensure that the learner’s concept image aligns with the formal concept definition. Tall and Vinner (1981) distinguished between the formal definition of a mathematical concept and the broader collection of images, associations and processes evoked by it. In the case of imaginary numbers, everyday language may form a misleading concept image before formal instruction begins.
Descriptive terms such as orthogonal component and quarter-turn operator can help reconstruct that image. They do not eliminate the historical terminology; they reveal the mathematical structure concealed beneath it.

10. CONCLUSION

The term imaginary number is historically established, but its ordinary-language meaning can mislead students into thinking that such numbers are fictitious, unreal or mathematically inferior to real numbers. This misunderstanding obscures the fact that complex numbers form a coherent two-dimensional number system and are indispensable in mathematics, physics, engineering and modern technology.

Will the official terminology change? Probably not. Terms such as real number, imaginary number and complex number are deeply embedded in textbooks, university curricula, research literature and scientific practice. A complete renaming would therefore be neither practical nor likely.

However, educators do not need to wait for an official change in terminology before improving the way complex numbers are taught. A practical approach is to retain the historical terms while introducing descriptive alternatives that reveal their geometric and operational meanings. For example:

An imaginary number—better understood geometrically as a rotational, orthogonal or vertical number—is represented along an axis perpendicular to the real-number, or linear-number, axis.

Similarly, real numbers may initially be described as linear or horizontal-axis numbers, while imaginary numbers may be introduced as vertical, orthogonal or rotational-axis numbers. These expressions should not be presented as formal replacements, but as pedagogical bridge terms that help learners form an accurate mental model before mastering the conventional vocabulary.

A diagram presenting linear numbers and vertical numbers side by side can help students understand the complex plane as a two-dimensional number system rather than as an artificial combination of "real" and "unreal" quantities. It also clarifies that a complex number, $\mathbf{z=a+bi}$, contains two mathematically legitimate components: one measured along the real axis and the other along a perpendicular axis.

Such an approach respects the historical continuity of mathematics while making its underlying structure more accessible. The established terminology may remain unchanged, but the explanation does not have to remain trapped in the past. By combining traditional terminology with clearer geometric descriptions, educators can help students understand complex numbers more quickly, accurately and intuitively.

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